AQI: Advanced Quantum Information
Lecture 7 (Module 2): State and Process Tomography
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Lecturer: Dr. Mark Tame

Introduction

We saw in the last lecture that if we're given just a single copy of an unknown state ρ, then it's impossible for us to distinguish it from other states if we don't know the measurement basis to use, or in the case it's from a set of non-orthonormal states then we have no hope for sure. The basic problem here is that there's no measurement that allows us to distinguish, say, the states $|0\rangle$ and $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. This also means that we're not able to characterise ρ faithfully with a single measurement. However, it is possible to estimate ρ if we're given a large enough number of copies of it. For instance in an experiment where we can repeat the experiment many times to produce many copies of ρ.

1 State tomography

Let's take a qubit density matrix ρ. Now the set $\{\frac{1}{\sqrt{2}}\mathbb{1}, \frac{1}{\sqrt{2}}X, \frac{1}{\sqrt{2}}Y, \frac{1}{\sqrt{2}}Z\}$ forms an orthonormal set of matrices with respect to the Hilbert-Schmidt inner product, $(A, B) = \mathrm{Tr}(A^\dagger B)$, where for an arbitrary operator acting on the Hilbert space we have

$$B = \sum_i \mathrm{Tr}(A_i^\dagger B) A_i \qquad c.f. \qquad |\psi\rangle = \sum_i \langle\psi_i|\psi\rangle\, |\psi_i\rangle\,. \tag{1}$$

A complete set has d^2 matrices, where we can choose $A_0 = \frac{1}{\sqrt{d}}\mathbb{1}$ and $\mathrm{Tr}(A_i A_j) = \delta_{ij}$. Orthogonality implies that each A_i for $i \neq 0$ is traceless, $\mathrm{Tr}(\mathbb{1}A_i) = \mathrm{Tr}(A_i) = 0$. Thus for a qubit we have $\mathbb{1}$ and the Pauli matrices, where one can write

$$\rho = \frac{1}{2}[\mathrm{Tr}(\rho)\mathbb{1} + \mathrm{Tr}(X\rho)X + \mathrm{Tr}(Y\rho)Y + \mathrm{Tr}(Z\rho)Z]. \tag{2}$$

Here, $\mathrm{Tr}(A\rho) = \langle A\rangle$ is the average value of the observable A. So we can estimate, say $\mathrm{Tr}(Z\rho)$, by measuring the observable Z a large number of times. Measuring Z m times we get the values $z_1, z_2, z_3, \ldots z_m$, where $z_i = \pm 1$. We then have $\mathrm{Tr}(Z\rho) = \sum_i \frac{z_i}{m}$ and via the central limit theorem this estimate (for large m) becomes Gaussian with mean equal to $\mathrm{Tr}(Z\rho)$ and standard deviation $\frac{\Delta(Z)}{\sqrt{m}}$, where $\Delta(Z)$ is the standard deviation for a single measurement of Z for which $\Delta(Z) \leq 1$.Therefore the standard deviation of our estimate of $\mathrm{Tr}(Z\rho)$ is at most $1/\sqrt{m}$. We can do the same for $\mathrm{Tr}(X\rho)$ and $\mathrm{Tr}(Y\rho)$ too. Note that $\mathrm{Tr}(\mathbb{1}\rho) = 1$ for a valid density operator. Thus we can obtain a good estimate for ρ given a large enough sample size.

In general for n qubits we have

$$\rho = \sum_{\underline{v}} 2^{-n}[\mathrm{Tr}(\sigma_{v_1} \otimes \sigma_{v_2} \ldots \sigma_{v_n}\rho)\sigma_{v_1} \otimes \sigma_{v_2} \ldots \sigma_{v_n}], \tag{3}$$

where the sum is over all vectors $\underline{v} = (v_1, \ldots, v_n)$, with $v_i \in \{0, 1, 2, 3\}$ and $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} := \{\mathbb{1}, X, Y, Z\}$.

An example of this general decomposition is the two-qubit state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where $\rho = |\phi^+\rangle\langle\phi^+|$. This has the decomposition

$$\rho = \mathbb{1} \otimes \mathbb{1} + X \otimes X - Y \otimes Y + Z \otimes Z. \tag{4}$$

One can easily generalise this to qudits by finding the correct orthonormal (and Hermitian) set of matrices for the relevant Hilbert space.

1.1 Example state tomography

Below is an example from an actual experiment with photons showing the elements of the reconstructed density matrix for the ideal state

$$|\phi_c\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{1234} \tag{5}$$

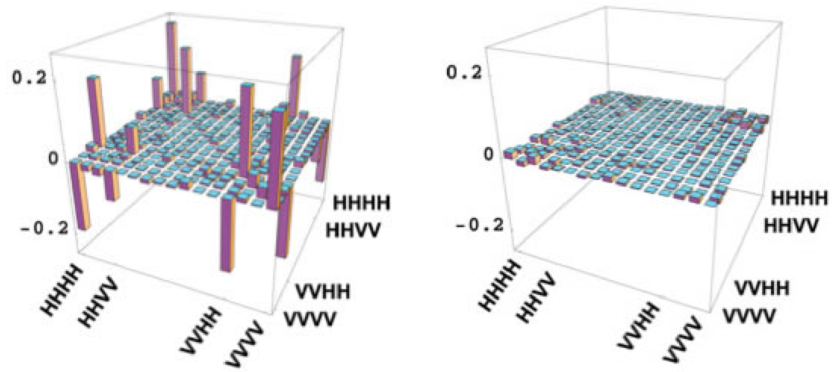

Figure 1: Taken from M. S. Tame *et al.* Phys. Rev. Lett. 98, 140501 (2007). Left hand side is the real part and right hand side is the imaginary part. Here the computational basis is represented by the polarization of the photons, $\{|0\rangle, |1\rangle\} := \{|H\rangle, |V\rangle\}$

2 Process tomography

What if we want to characterise the dynamics of a quantum system? This is known as system identification in classical systems and for quantum systems it's called quantum process tomography. Consider the physical process acting on the quantum system ρ described by the channel $\mathcal{E}$

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \qquad \sum_i E_i^\dagger E_i = \mathbb{1}. \tag{6}$$

We'd like to know the form of this Kraus decomposition based on experimentally measurable observables and therefore find the E_i's. To do this we use a fixed set of operators $\tilde{E}_i$ which form a basis for the set of operators

$$E_i = \sum_m \mathrm{Tr}(\tilde{E}_m^\dagger E_i)\tilde{E}_m = \sum_m e_{im}\tilde{E}_m \tag{7}$$

which gives

$$\mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \rho \tilde{E}_n^\dagger \chi_{mn}, \qquad \text{with} \qquad \chi_{mn} = \sum_i e_{im} e_{in}^*. \tag{8}$$

This decomposition means that the channel $\mathcal{E}$ can be completely described by a complex number matrix χ and a fixed set of operators $\{\tilde{E}_m\}$. So we just need to find a way to obtain the values of the entries in the χ matrix. The procedure for this is outlined below.

Consider a fixed, linearly independent basis for the space of $d \times d$ matrices: $\rho_j, 1 \leq j \leq d^2$. For example, the set of operators $|n\rangle\langle m|$. The output state of the channel $\mathcal{E}$ acting on one of these inputs $\mathcal{E}(|n\rangle\langle m|)$ can be found by preparing the following input states

$$|n\rangle\,, |m\rangle\,, |+\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |m\rangle) \text{ and } |+_y\rangle = \frac{1}{\sqrt{2}}(|n\rangle + i\,|m\rangle), \tag{9}$$

then forming linear combinations of the outputs $\mathcal{E}(|n\rangle\langle n|)$, $\mathcal{E}(|m\rangle\langle m|)$, $\mathcal{E}(|+\rangle\langle +|)$ and $\mathcal{E}(|+_y\rangle\langle +_y|)$ as

$$\mathcal{E}(|n\rangle\langle m|) = \mathcal{E}(|+\rangle\langle +|) + i\mathcal{E}(|+_y\rangle\langle +_y|) - \frac{1+i}{2}\mathcal{E}(|n\rangle\langle n|) - \frac{1+i}{2}\mathcal{E}(|m\rangle\langle m|). \tag{10}$$

Thus we can find $\mathcal{E}(\rho_j)$ for each ρ_j by performing state tomography on the output states of the above four input states. Actually, we could stop here! But we want to recover the Kraus decomposition as it's a more powerful description.

Note that we could also write

$$\mathcal{E}(\rho_j) \;\; = \;\; \sum_k \lambda_{jk} \rho_k, \tag{11}$$

as the ρ_k are basis states ρ_j (just with a different index). Here the $\mathcal{E}(\rho_j)$ are experimentally determined, the ρ_k are fixed beforehand and the λ_{jk} can be calculated once the $\mathcal{E}(\rho_j)$ are known. Thus we can write

$$\begin{aligned} \mathcal{E}(\rho_j) &= \sum_{mn} \tilde{E}_m \rho_j \tilde{E}_n^\dagger \chi_{mn} = \sum_{mnk} \beta_{jk}^{mn} \rho_k \chi_{mn} \\ &= \sum_k \lambda_{jk} \rho_k, \end{aligned} \tag{12}$$

where we define $\tilde{E}_m \rho_j \tilde{E}_n^\dagger = \sum_k \beta_{jk}^{mn} \rho_k$. Thus equating the last term of Eq. (12) on the first line and the second line of Eq. (12) we have

$$\lambda_{jk} = \sum_{mn} \beta_{jk}^{mn} \chi_{mn}. \tag{13}$$

Why did we do this? Well, the λ_{jk} are calculated once the $\mathcal{E}(\rho_j)$ are experimentally determined (see Eq. (11)) and the β_{jk}^{mn} elements are set once $\{\tilde{E}_m\}$ and $\{\rho_j\}$ are set, so that the χ_{mn} elements can be found once λ_{jk} are known and the β_{jk}^{mn} are set by inverting Eq. (13) as

$$\chi_{mn} = \sum_{jk} \kappa_{jk}^{mn} \lambda_{jk}. \tag{14}$$

This can be done on the computer if the system is very large, or by hand if manageable. Here, κ is the generalised inverse of β, *i.e.* it satisfies the following relation

$$\beta_{jk}^{mn} = \sum_{st\,xy} \beta_{jk}^{st} \kappa_{st}^{xy} \beta_{xy}^{mn}. \tag{15}$$

Thus, we have a way to find the χ_{mn} elements and we have the fixed set $\{\tilde{E}_m\}$. Now we'd like to find the $\{E_i\}$ in the original Kraus decomposition of Eq. (6). We know that χ is a positive Hermitian matrix, therefore it can be diagonalised using $D = U^\dagger \chi U$, or written another way

$$D = U^\dagger \chi U \rightarrow \chi_{mn} = \sum_{ij} U_{mj} d_j \delta_{ji} U^*_{ni}, \tag{16}$$

where $d_j \delta_{ji} = D_i$ are positive and real. Making the association $e_{im} = \sqrt{D_i} U_{mi}$ and $e^*_{in} = \sqrt{D_i} U^*_{ni}$ we have from Eqs. (7) and (8)

$$E_i = \sum_m e_{im} \tilde{E}_m = \sum_m \sqrt{D_i} U_{mi} \tilde{E}_m. \tag{17}$$

Thus, we have found the Kraus decomposition for the channel $\mathcal{E}(\rho)$.

Ok, so in summary quantum process tomography follows these steps:

- A λ matrix is found from state tomography once $\{\rho_j\}$ is chosen.
- A χ matrix is found from the λ matrix via β, once $\{\tilde{E}_m\}$ is chosen.
- The set $\{E_i\}$ for the Kraus decomposition is found once χ is known.

2.1 Example process tomography

Consider the channel

$$\$: \rho \rightarrow \rho' = \mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \tag{18}$$

where ρ is a single qubit.

- Choose a fixed set: $\tilde{E}_0 = \mathbb{1}, \tilde{E}_1 = X, \tilde{E}_2 = -iY, \tilde{E}_3 = Z$. Note that although the operators in this set are not normalised (with respect to the Hilbert-Schmidt inner product), this is taken into account later by the χ matrix.
- Choose the fixed basis $\{\rho_j\} = \{\rho_1, \rho_2, \rho_3, \rho_4\} = \{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\}$.
- Prepare the input states $\{|0\rangle, |1\rangle, |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |+_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\}$ and determine the output states $\mathcal{E}(|0\rangle\langle 0|), \mathcal{E}(|1\rangle\langle 1|), \mathcal{E}(|+\rangle\langle +|)$ and $\mathcal{E}(|+_y\rangle\langle +_y|)$ via state tomography.

Below is an example from a photonic experiment showing the input states and output states.

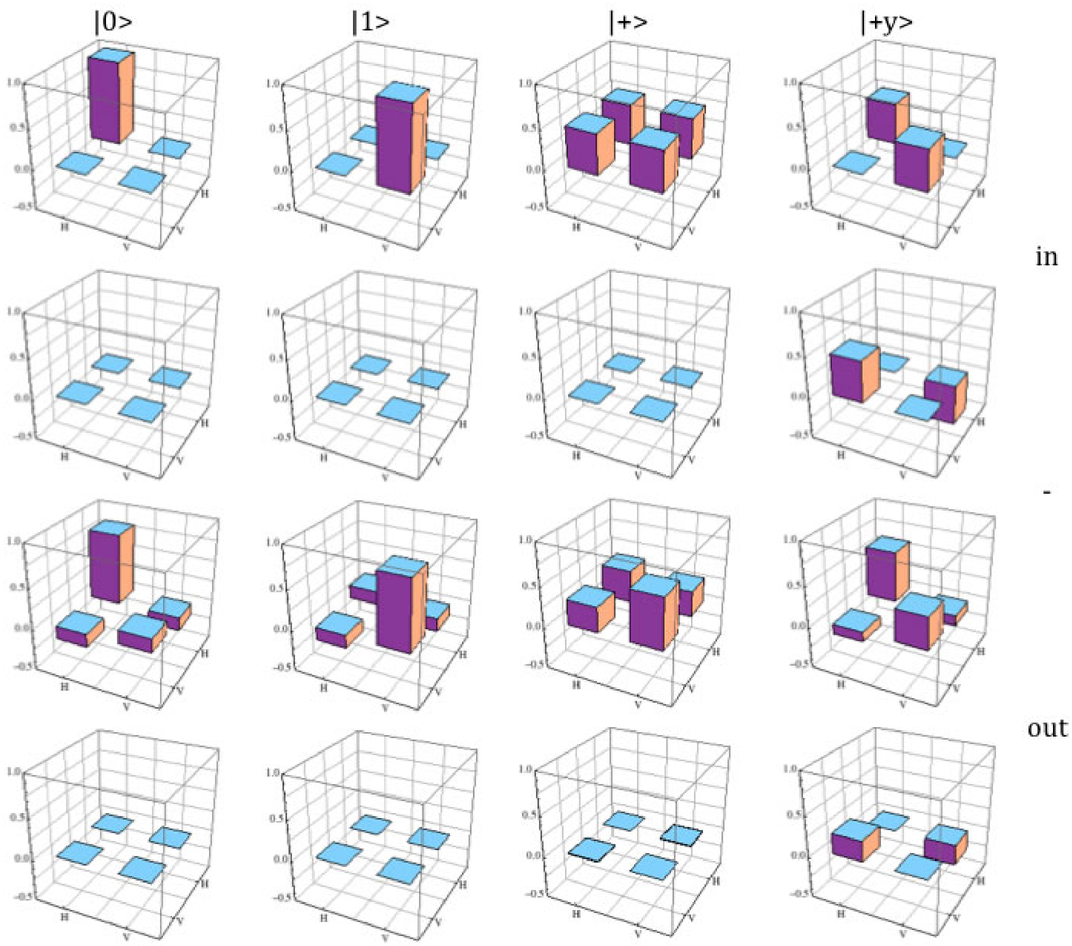

Figure 2: Taken from R. Prevedel *et al.* Phys. Rev. Lett. 99, 250503 (2007). The first row is the real part and the second row is the imaginary part for the input (in) and output (out) states respectively.

- From the output states we can then write

$$
\begin{aligned}
\rho_1' &= \mathcal{E}(|0\rangle\langle 0|) \qquad (19)\\
\rho_2' &= \mathcal{E}(|0\rangle\langle 1|) = \mathcal{E}(|+\rangle\langle +|) + i\mathcal{E}(|+_y\rangle\langle +_y|) - \frac{1+i}{2}(\mathcal{E}(|0\rangle\langle 0|) + \mathcal{E}(|1\rangle\langle 1|))\\
\rho_3' &= \mathcal{E}(|1\rangle\langle 0|) = \mathcal{E}(|+\rangle\langle +|) - i\mathcal{E}(|+_y\rangle\langle +_y|) - \frac{1-i}{2}(\mathcal{E}(|0\rangle\langle 0|) + \mathcal{E}(|1\rangle\langle 1|))\\
\rho_4' &= \mathcal{E}(|1\rangle\langle 1|)
\end{aligned}
$$

- This gives us the matrix λ (from Eq. (11)) and with the tensor β (from our chosen set $\{\tilde{E}_m\}$) we can calculate the matrix χ. Due to the choice of $\{\tilde{E}_m\}$ and $\{\rho_j\}$ we have

$$
\beta = \begin{pmatrix} \mathbb{1}\otimes\mathbb{1} & \mathbb{1}\otimes X & \mathbb{1}\otimes iY & \mathbb{1}\otimes Z \\ X\otimes\mathbb{1} & X\otimes X & X\otimes iY & X\otimes Z \\ iY\otimes\mathbb{1} & iY\otimes X & iY\otimes iY & iY\otimes Z \\ Z\otimes\mathbb{1} & Z\otimes X & Z\otimes iY & Z\otimes Z \end{pmatrix}. \qquad (20)
$$

- Then using $\lambda_{jk} = \sum_{mn} \beta_{jk}^{mn} \chi_{mn}$, where $\mathcal{E}(\rho_j) = \rho'_j = \sum_k \lambda_{jk}\rho_k$ defines the λ_{jk}, *e.g.*

$$\rho'_1 = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{13} & \lambda_{14} \end{pmatrix}, \tag{21}$$

$$\rho'_2 = \begin{pmatrix} \lambda_{21} & \lambda_{22} \\ \lambda_{23} & \lambda_{24} \end{pmatrix},$$

(and so on for ρ'_3 and ρ'_4), one finds

$$\chi = \Lambda \begin{pmatrix} \rho'_1 & \rho'_2 \\ \rho'_3 & \rho'_4 \end{pmatrix} \Lambda, \tag{22}$$

where

$$\Lambda = \frac{1}{2} \begin{pmatrix} \mathbb{1} & X \\ X & -\mathbb{1} \end{pmatrix}. \tag{23}$$

- Now that we have χ we have completely characterised the channel $\mathcal{E}$, and with a few more steps we can have the Kraus decomposition!

References

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